Claremont Colleges

Scholarship @ Claremont

HMC Senior Theses

HMC Student Scholarship

2020

An Exploration of Combinatorial Interpretations for Fibonomial Coefficients

Richard Shapley

An Exploration of Combinatorial Interpretations for Fibonomial Coefficients

Ricky Shapley

Arthur T. Benjamin, Advisor

Curtis Bennett, Reader

HARVEY MUDD COLLEGE

Department of Mathematics

May, 2020

Abstract

We can define Fibonomial coefficients as an analogue to binomial coefficients as $\binom{n}{k}_F = \frac{F_n \cdot F_{n-1} \cdots F_{n-k+1}}{F_k \cdot F_{k-1} \cdots F_1}$, where F_n represents the nth Fibonacci number. Like binomial coefficients, there are many identities for Fibonomial coefficients that have been proven algebraically. However, most of these identities have eluded combinatorial proofs.

Sagan and Savage (2010) first presented a combinatorial interpretation for these Fibonomial coefficients. More recently, Bennett et al. (2018) provided yet another interpretation, that is perhaps more tractable. However, there still has been little progress towards using these interpretations of the Fibonomial coefficient to prove any of the identities.

Within this thesis, I seek to explore both proofs for Fibonomial identities that have yet to be explained combinatorially, as well as potential alternatives to the thus far proposed combinatorial interpretations of Fibonomial coefficients themselves.

Contents

Abstract **iii**

Acknowledgments **ix**

1 Background **1**
1.1 Generalizations of Binomial Coefficients 1
1.2 Tilings 3
1.3 Interpretations of Fibonomial Coefficients 6

2 A Modified Interpretation **11**
2.1 Motivation 11
2.2 The Interpretation 12
2.3 Proof of the Interpretation 14

3 Some Combinatorial Proofs of Fibonomial Identities **17**

4 Some Analysis of the Combinatorial Interpretations **21**
4.1 Correspondence with the Rectangular Interpretation 21
4.2 Extension Into Three Dimensions 23

5 A Brief Exploration of Alternating Sum Identities **27**

6 Future Work **33**

Bibliography **35**

List of Figures

1.1 Tilings on a $1 \times n$ grid. 4
1.2 Circular tilings on a $1 \times n$ grid. 5
1.3 Tilings of a lattice path in a rectangular grid, for $n = 4, m = 3$ 6
1.4 A pyramid grid for $n = 7$. 8
1.5 A lattice path. From the bottom, the steps in this path are "I", "L", "L", "I", "L", "I", "I". 8
1.6 One partial tiling for $n = 7$ and $k = 3$. The unshaded regions are not tiled. 9

2.1 A symmetric pyramid diagram for $n = 7$. 12
2.2 A path through the symmetric pyramid. From the bottom, the steps in this path are "right", "left", "left", "right", "left", "right", "right". 13
2.3 One partial circular tiling for $n = 7$ and $k = 3$. The unshaded regions are not tiled. Note that in the top two rows, we are tiling 0 squares, which can be done 2 ways each. 13

4.1 Partial Tilings of $\binom{5}{2}_F$ with exactly 6 squares tiled. 22
4.2 A tiled lattice path and the corresponding partial tiling. . . . 23

Acknowledgments

I’d like to thank my advisor Arthur Benjamin for his support and encouragement and also my second reader Curtis Bennett.

Chapter 1

Background

1.1 Generalizations of Binomial Coefficients

Let us begin this exploration with some revision of well known concepts. We will start with *binomial coefficients*.

Definition 1.1. The binomial coefficient

$$\binom{n}{k} = \frac{n \cdot (n-1) \cdots (n-k+1)}{k \cdot (k-1) \cdots 1}$$

While this algebraic definition is useful, binomial coefficients are also frequently considered combinatorially. In particular, the binomial coefficient $\binom{n}{k}$ counts the number of ways to choose k unordered elements from a set of n elements. This is a great example of how algebraic objects can be imbued with some combinatorial meaning.

However, we can move beyond binomial coefficients to a more generalized structure. While it is difficult to imagine how we might generalize the combinatorial interpretation, we can easily manipulate the algebraic definition to extend binomial coefficients to *generalized binomial coefficients*.

Definition 1.2. The generalized binomial coefficient

$$\binom{n}{k}_U = \frac{U_n \cdot U_{n-1} \cdots U_{n-k+1}}{U_k \cdot U_{k-1} \cdots U_1}, \qquad \binom{n}{0}_U = 1$$

where U is a sequence of numbers.

Different authors may place different restrictions on U to achieve various results. Some require that U be a sequence of positive integers, or contain no 0 values. Others require that the elements of U satisfy some kind of recurrence, or are solutions to a specific set of equations. For now, we will leave this definition in this most general form.

Regardless of how U is defined, we can clearly see the analogues between these generalized binomial coefficients, and the original binomial coefficients. In fact, we arrive at the original binomial coefficients when we allow U to be the sequence $\{1, 2, 3, \dots\}$.

However, these generalized binomial coefficients are not so useful combinatorially, because we have no guarantee that any coefficient is an integer, or that it even exists. Therefore, we will consider a subset of generalized binomial coefficients based on Lucas sequences, and a particular case of these which we will refer to as Fibonomials.

Definition 1.3. A Lucas sequence of the first kind W[1] is a sequence of integers generated as follows:

Given two integers, s and t, let $W_0 = 0$, $W_1 = 1$, and

$$W_n = sW_{n-1} + tW_{n-2}.$$

Now, when we use W as the sequence for generalized binomial coefficients, we arrive at what we can call *Lucasnomial coefficients*. One perhaps surprising feature of these Lucasnomial coefficients is that they are guaranteed to have integer values, a good sign if we wish to find some combinatorial meaning for them.

Since polynomials can be difficult to deal with, we will now examine a specific case of Lucas sequences. When we let $s = 1$ and $t = 1$, we arrive at the sequence $F_0 = 0$, $F_1 = 1$, $F_2 = 1$, $F_3 = 2$, etc. The choice of variable for this sequence was deliberate, as these are indeed the Fibonacci numbers! The natural next step is to consider what happens when we use the Fibonacci sequence in our generalized binomial coefficients.

Definition 1.4. The Fibonomial coefficient

$$\binom{n}{k}_F = \frac{F_n \cdot F_{n-1} \cdots F_{n-k+1}}{F_k \cdot F_{k-1} \cdots F_1}$$

[1]Here, we use W to avoid confusion with the sequence L that we will later use to describe the sequence of Lucas numbers.

We have already indicated that Fibonomial coefficients must have integer values, and hence are likely candidates for combinatorial interpretation. But for now, we will stick with our algebraic definitions. To elucidate this definition of Fibonomial coefficients, here are a few examples.

$$\binom{10}{0}_F = 1$$
$$\binom{10}{10}_F = 1$$
$$\binom{10}{1}_F = F_{10} = 55$$
$$\binom{10}{3}_F = \frac{F_{10} \cdot F_9 \cdot F_8}{F_3 \cdot F_2 \cdot F_1} = \frac{55 \cdot 34 \cdot 21}{2 \cdot 1 \cdot 1} = 19635$$

Finally, just as Pascal's triangle holds the binomial coefficients, we can construct a similar table for Fibonomial coefficients. I have produced the first several rows below.

$$\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 1 \quad 1 \\
1 \quad 2 \quad 2 \quad 1 \\
1 \quad 3 \quad 6 \quad 3 \quad 1 \\
1 \quad 5 \quad 15 \quad 15 \quad 5 \quad 1 \\
1 \quad 8 \quad 40 \quad 60 \quad 40 \quad 8 \quad 1
\end{array}$$

1.2 Tilings

Here, we will discuss some well known combinatorial interpretations of Fibonacci numbers and related sequences, primarily focusing on tilings.

Let f_n be the number of ways to tile a $1 \times n$ grid with squares and dominoes. (Here, when we say tile, we mean that we wish to place squares (monominoes) and dominoes on the grid such that every cell of the grid is covered.) After some brief examination (see Figure 1.1), it appears that the f_n follow the Fibonacci numbers.

Theorem 1.1. *For $n \geq 0$, $f_n = F_{n+1}$*

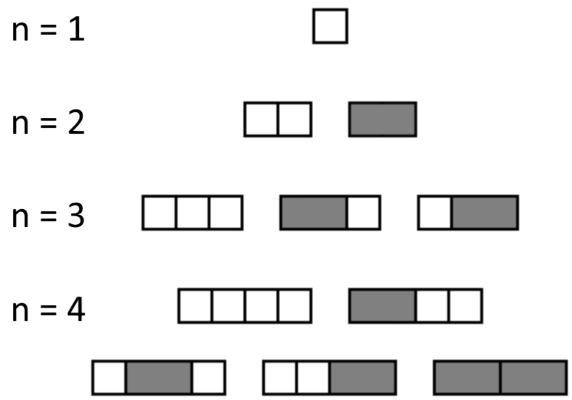


Figure 1.1 Tilings on a $1 \times n$ grid.

Proof. We can prove this inductively by considering the last tile and counting how many ways there are to tile the remaining grid. □

Although this interpretation is specifically for Fibonacci numbers, we can extend this tiling interpretation to handle any Lucas sequence. Given integers s and t and some tiling T, let

$$w(T) = s^m t^d$$

where m is the number of monominoes in the tiling, and d is the number of dominoes in the tiling. Then, if we let τ_n be the set of all tilings of a $1 \times n$ grid, define

$$w_n = \sum_{T \in \tau_n} w(T).$$

We can think of $w(T)$ as the weight of a tiling. If the weight of a square is s and the weight of a domino is t, then the total weight of a specific tiling is the product of the weights of all the tiles that make up the tiling. Then w_n represents the sum of the weights of all tilings of the appropriate length. With some examination, we can see that these w_n follow the pattern of the Lucas sequences we defined earlier. So it is not surprising that the following is true.

Theorem 1.2. *For* $n \geq 0$, $w_n = W_{n+1}$

Proof. Again, prove inductively by considering the last tile. □

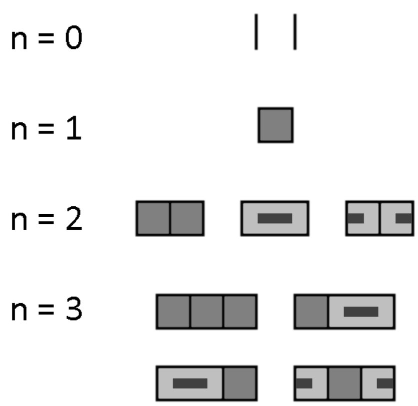


Figure 1.2 Circular tilings on a $1 \times n$ grid.

In addition to the traditional tilings that we have described, we can also consider circular tilings, sometimes also called bracelet tilings.

Let c_n be the number of ways to tile a $1 \times n$ grid with squares and dominoes, where we are also allowed to use a single domino to cover the first and last squares in the grid. The values of c_n are displayed for the first few n in Figure 1.2. For reasons that will soon become clear, we choose to define $c_0 = 2$.

At this point, it may be useful to define the Lucas numbers. This is another famous sequence of numbers that are closely related to the Fibonacci numbers. Curiously, they do not form a Lucas sequence of the first kind[2]. Define the Lucas numbers L_n by $L_0 = 2$, $L_1 = 1$, and recursively let

$$L_n = L_{n-1} + L_{n-2}.$$

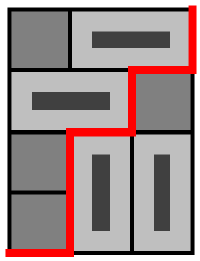
Then the first few Lucas numbers are $2, 1, 3, 4, 7, 11, \ldots$. Notice that the numbers of circular tilings c_n correspond exactly with the Lucas numbers L_n.

Theorem 1.3. *For $n \geq 0$, $c_n = L_n$*

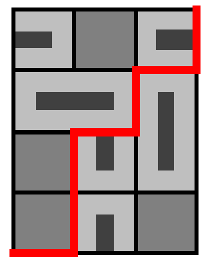
Proof. As before, we can prove this inductively by considering the last tile. $\square$

It is because of this correspondence that we claimed $c_0 = 2$. Other definitions may completely circumvent this issue by defining circular tilings

[2]The Lucas numbers actually form a Lucas sequence of the second kind.



a. Standard tiling



b. Circular tiling

Figure 1.3 Tilings of a lattice path in a rectangular grid, for $n = 4, m = 3$

only for $n \geq 1$, however, I have chosen to use this slightly uncomfortable definition at $n = 0$, as we can then make use of it in some combinatorial interpretations.

1.3 Interpretations of Fibonomial Coefficients

We will now move to some interpretations of Fibonomial coefficients that others have developed. Although there are a few other interpretations, here we will only look at the combinatorial interpretations presented by Sagan and Savage, and the one described by Bennett et al.

1.3.1 Fibonomial Coefficients as Tilings in a Rectangular Grid

We will begin by looking at the two combinatorial interpretations proposed by Sagan and Savage.

Both interpretations by Sagan and Savage (2010) subtly take advantage of lattice paths. Consider an $n \times m$ grid. Notice that $\binom{n+m}{n}$ is the number of lattice paths from the bottom left corner to the top right, using only unit steps up or to the right.

Here is an interpretation for $\binom{n+m}{n}_F$. Start with a $n \times m$ grid, and create a lattice path from the bottom left corner to the top right corner, using only steps in the upward or rightward directions. This lattice path will divide the grid into two regions: the region above the path, and the region below the path. In the region above the path, tile each row with dominoes and monominoes in the standard fashion. In the region below the path, tile each column with dominoes and monominoes, with the added restriction that tilings cannot begin with a monomino. Figure 1.3a has an example of one such tiling. The total number of ways to tile all possible lattice paths is enumerated by $\binom{n+m}{n}_F$.

A second interpretation by Sagan and Savage is very similar. Beginning with the same grid and lattice paths, we will just tile the regions differently. In the region above the path, cover each row with dominoes and monominoes in a circular tiling. And in the region below the path use a circular tiling to cover each column with dominoes and monominoes. An example of this is in Figure 1.3b. The total number of ways to tile all possible lattice paths in this manner is enumerated by $2^{n+m}\binom{n+m}{n}_F$.

We can prove both of these interpretations by using the recursive relationships that defines the Fibonomial coefficients. Sagan and Savage (2010) showed[3] that for $m, n \geq 1$,

$$\binom{m+n}{m}_F = F_{n+1}\binom{m+n-1}{m-1}_F + F_{m-1}\binom{m+n-1}{n-1}_F \tag{1.1}$$

and

$$\binom{m+n}{m}_F = \frac{L_n}{2}\binom{m+n-1}{m-1}_F + \frac{L_m}{2}\binom{m+n-1}{n-1}_F. \tag{1.2}$$

Sagan and Savage were then able to show that their interpretations satisfied these recurrences, and so were necessarily correct.

Also note that both of these interpretations can be generalized to Lucasnomial coefficients by calculating the weights of each tiling. In this case, the Lucasnomial coefficients would indicate the sum of the weights of all tilings for all lattice paths of the grid.

1.3.2 Fibonomial Coefficients as Tilings in a Pyramid

Now, we will examine the interpretation proposed by Bennett et al. (2018).

Begin with a pyramid-shaped grid of height n, as shown in Figure 1.4. Consider a point p on the bottom of the pyramid that is k units from the bottom left corner. Create a lattice path from p to the top of the pyramid, consisting of only "I" shaped steps and "L" shaped steps. An "I" step moves up a single unit, and an "L" step first moves left one unit, then up one unit. One such path is shown in Figure 1.5.

In rows where the lattice path is comprised of just an "I" segment (as opposed to an "L" segment), cover the squares to the left of the "I" with

[3]They actually proved this for more general Lucasnomial coefficients, but we only need this specific case.

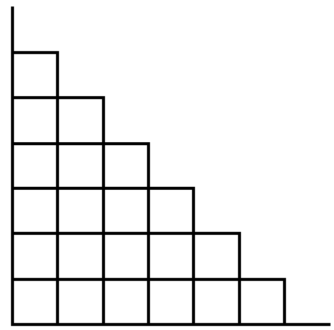


Figure 1.4 A pyramid grid for $n = 7$.

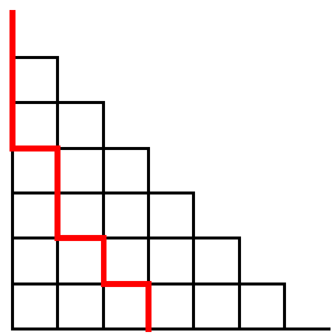


Figure 1.5 A lattice path. From the bottom, the steps in this path are "I", "L", "L", "I", "L", "I", "I".

dominoes and monominoes. And for rows where the lattice path instead has an "L" segment, cover the squares to the right of the "L" with dominoes and monominoes, with the added restriction that the tiling cannot begin with a monomino. This tiling, in which we cover part of every row, we will call a partial tiling. An example of one such partial tiling is in Figure 1.6. Enumerating all partial tilings for all lattice paths from this point p to the top of the pyramid gives $\binom{n}{k}_F$.

The proof for this interpretation is surprisingly intuitive. Completely tiling every row of the diagram independently would allow for $F_1 \cdot F_2 \cdots F_n$ possible tilings. But in our partial tilings, notice that to the left of the path, we always avoid tiling grids from size 1×0 to $1 \times (k-1)$, and similarly on the right of the path, we avoid tiling grids of size 1×0 to $1 \times (n-k-1)$. So

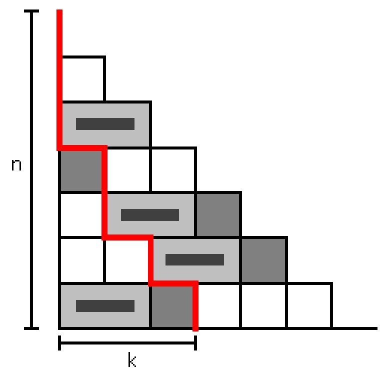


Figure 1.6 One partial tiling for $n = 7$ and $k = 3$. The unshaded regions are not tiled.

to count the number of partial tilings, we divide by the number of ways to tile the untiled regions, which gives us exactly the Fibonomial coefficient.

As before, this combinatorial interpretation for Fibonomial coefficients can easily be extended to Lucasnomial coefficients. Just as in the earlier interpretations, instead of simply counting all the tilings, we calculate the weight of each tiling. The sum of these weights is equal to the corresponding Lucasnomial coefficient.

It may not immediately be clear why we might prefer this last combinatorial interpretation of the Fibonomial coefficients. But when examining the proofs of the different interpretations, it emerges that the interpretation by Bennett et al. is more natural. The proof for Sagan and Savage's tilings of lattice paths through rectangular grids relies on showing that the tilings satisfy the same recurrence relation as the Fibonomial coefficients. But the proof by Bennett et al. that partial tilings enumerate Fibonomial coefficients is direct. It describes how the structures satisfy the algebraic definition of the Fibonomial coefficient, instead of relying on the recurrence relation.

Therefore, it is with this great hope that I set out to prove some Fibonomial identities using the combinatorial interpretation of Bennett et al.

Chapter 2

A Modified Interpretation

2.1 Motivation

With a largely unexplored new interpretation of the Fibonomial coefficients, and our goal to prove identities, the most obvious next step is to begin attacking some identities with this interpretation. For identities in which some Fibonomial expression should equal another, we might naturally begin by looking at small examples. And if we can use this interpretation to prove the identities, we should be able to find a bijection between the partial tilings of the pyramid. But these bijections were not forthcoming, and I had reason to believe that if I could find bijections, they would be very complicated.

A couple of Fibonomial identities were already proven combinatorially by Bennett et al. using their proposed method. One of the identities was the symmetric identity,

$$\binom{n}{k}_F = \binom{n}{n-k}_F.$$

While this identity appears to be one of the simplest possible Fibonomial identities, it was not so easy to prove. In fact, the provided proof created a clever but complicated algorithm for how to find the bijection between the partial tiling representations. The length of the proof for this simple identity was discouraging for finding elegant proofs for other identities.

Therefore, to hopefully inspire some simpler proofs, I found another combinatorial interpretation for the Fibonomial coefficients.

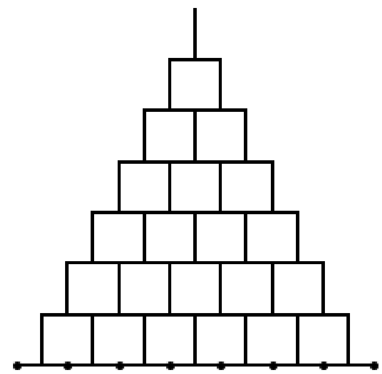


Figure 2.1 A symmetric pyramid diagram for $n = 7$.

2.2 The Interpretation

This interpretation is a synthesis of two interpretations that were described earlier. In short, we create partial tilings of a pyramid, but instead of using standard tilings, we use circular tilings. Begin with a pyramid-shaped diagram of height n, as shown in Figure 2.1. Notice that the pyramid has changed from an earlier version: I have shifted all the rows so that the pyramid is symmetric! This change was made to illustrate the symmetric nature of the interpretation. It also means that some of the language used to describe the method must also change, but we will manage.

Begin at a point that is k units from the bottom left corner. From here on, we will abuse this notation a bit, and call this point k. Create a path from k to the top of the pyramid, consisting of only "left" steps and "right" steps. We will define a "left" step as a step that travels half a unit to the left before following the first line up one unit. Similarly, a "right" step first travels right, before traveling one unit up at the first opportunity. One such path is shown in Figure 2.2.

In rows where the lattice path consists of just a "right" segment, cover the squares to the left of the path with dominoes and monominoes in a circular tiling. And for rows where the path has an "left" segment, cover the squares to the right of the path with dominoes and monominoes, again tiling circularly. Notice that this time, there is no restriction on when a tiling cannot begin with a monomino. However, we still have a strange feature. Recall that we defined the number of circular tilings of a 1×0 grid, $c_0 = 2$.

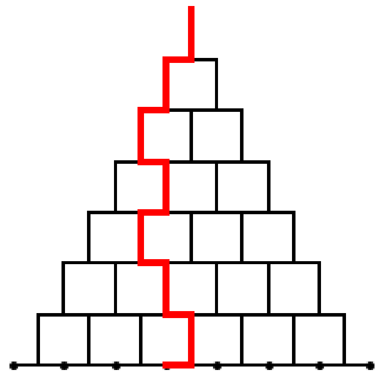


Figure 2.2 A path through the symmetric pyramid. From the bottom, the steps in this path are "right", "left", "left", "right", "left", "right", "right".

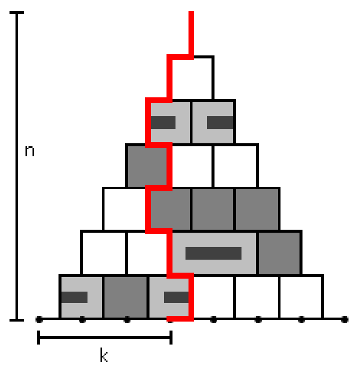


Figure 2.3 One partial circular tiling for $n = 7$ and $k = 3$. The unshaded regions are not tiled. Note that in the top two rows, we are tiling 0 squares, which can be done 2 ways each.

So whenever we are tiling nothing, we must actually add a factor of two to the number of tilings of the path.

As these are analogous to the partial tilings described earlier, we will call these *partial circular tilings*. An example of one such partial circular tiling is in Figure 2.3. Enumerating all partial circular tilings for paths beginning at k to the top of the pyramid gives $2^n \binom{n}{k}_F$, as shown below.

2.3 Proof of the Interpretation

Theorem 2.1. *For $n, k \geq 0$, enumerating all partial circular tilings for paths beginning at k in a pyramid of height n yields $2^n \binom{n}{k}_F$.*

Proof. We begin with equation 1.2. For $m, n \geq 1$,

$$\binom{m+n}{m}_F = \frac{L_n}{2}\binom{m+n-1}{m-1}_F + \frac{L_m}{2}\binom{m+n-1}{n-1}_F.$$

We can perform some simple algebra to arrive at an arguably friendlier version: For $0 < k < n$,

$$\binom{n}{k}_F = \frac{L_k}{2}\binom{n-1}{n-k-1}_F + \frac{L_{n-k}}{2}\binom{n-1}{k-1}_F. \tag{2.1}$$

As this recurrence can be used to generate all Fibonomial coefficients, if we can show that our interpretation satisfies this recurrence, along with some initial conditions, then we will have proven that the interpretation is correct.

First, let's handle the initial conditions. We desire the initial conditions $\binom{n}{0}_F = \binom{n}{n}_F = 1$. Thus, for interpretation, we must show that the number of partial circular tilings for paths beginning at 0 and the number of partial circular tilings for paths beginning at n are both equal to 2^n.

For each of these interpretations, where we begin at 0 or n, there is only one path to the top, which stays on the edge of the pyramid. This means we only need to consider the number of tilings of this one path. For every row, we are attempting to circularly tile an empty grid, as the region we attempt to tile is always towards the outside of the pyramid. We stated that there are two ways to do this, so in total, there are 2^n total partial circular tilings in each of the two cases. And we have shown that we meet our desired initial conditions.

Now, all that remains is to show that the interpretation follows the recurrence. Consider the number of partial circular tilings beginning at k. We have two cases: the first path step can either be a "right" step or a "left" step.

If it is a "left" step, then there are $n-k$ squares to tile circularly. We can tile these squares in $c_{n-k} = L_{n-k}$ ways. The number of ways to tile the rest of the pyramid is the number of partial circular tilings for a pyramid of height $n-1$, beginning at $k-1$.

If it is a "right" step, then there are k squares to tile circularly, which can be done in $c_k = L_k$ ways. Tiling the rest of the pyramid is simply the number of partial circular tilings for a pyramid of height $n-1$, beginning at k. But since the interpretation is symmetric, we can flip the pyramid, and discover that this is the same as the number of partial circular tilings for a $n-1$ pyramid, beginning at $n-1-k$.

So considering both cases for the bottom step provides the relationship

$$2^n \binom{n}{k}_F = 2^{n-1} L_k \binom{n-1}{n-k-1}_F + 2^{n-1} L_{n-k} \binom{n-1}{k-1}_F.$$

After we factor out the 2^n, we arrive at exactly Equation 2.1, as desired. $\square$

Chapter 3

Some Combinatorial Proofs of Fibonomial Identities

Well, we now have yet another combinatorial interpretation for Fibonomial coefficients. And I would like to use this to prove some identities.

The interpretation just introduced has one great benefit: it is symmetric. "Left" steps and their associated tilings are the exact mirror of "right" steps, which means that flipping any partial circular tiling horizontally will yield another partial circular tiling.

Theorem 3.1. *For $n, k \geq 0$,*

$$2^n \binom{n}{k}_F = 2^n \binom{n}{n-k}_F.$$

Proof. We observe that for every partial circular tiling beginning at k, we can obtain a partial circular tiling beginning at $n-k$ by taking the mirror image. □

This identity has already been proven combinatorially by Reiland (2011), and the equivalent $\binom{n}{k}_F = \binom{n}{n-k}_F$ by Bennett et al. (2018). But this proof is by far the simplest, and obviously reflected in the structure of the combinatorial interpretation.

We will now move on to some less obvious identities that we can prove using this interpretation.

Theorem 3.2. *For $n \geq 0$,*

$$\sum_{k=0}^{n+1} \binom{n+1}{k}_F = \sum_{j=0}^{n} L_j \binom{n}{j}_F.$$

Although I have not found any reference to this identity, we can easily derive it by summing over values of m in Equation 1.2.

Proof. We begin by multiplying through by 2^{n+1} to arrive at

$$\sum_{k=0}^{n+1} 2^{n+1}\binom{n+1}{k}_F = \sum_{j=0}^{n} 2L_j 2^n \binom{n}{j}_F.$$

Now we will prove combinatorially.

The left side counts how many partial circular tilings there are in a pyramid of height $n+1$ from all starting points.

There are $n+1$ different points the path could pass through on the second-to-last row. If we index these points from left to right, they will run from 0 to n. Each of these points can be reached in two ways from the bottom row.

If a point j is reached by a "right" step, then we will have to tile j squares to the left on the bottom row, which can be done in L_j ways. We can tile the rest of the pyramid by counting partial circular tilings beginning at j in a pyramid size n. This gives us a term of $\sum_{j=0}^{n} L_j 2^n \binom{n}{j}_F$.

If j is instead reached by a "left" step, then we will need to tile $n-j$ squares to the right on the bottom row, done in L_{n-j} ways. The number of ways the other rows of the pyramid can be tiled is the number of partial circular tilings beginning at j in a pyramid size n. But by symmetry, this is the same as the number of partial circular tilings beginning at $n-j$. So this gives us a term $\sum_{j=0}^{n} L_{n-j} 2^n \binom{n}{n-j}_F$.

Reversing the indices on the second sum to combine the sums gives that the number of partial circular tilings in a $n+1$ pyramid is $\sum_{j=0}^{n} 2L_j 2^n \binom{n}{j}_F$. □

Following are a class of identities obtained by fixing the bottom term in the Fibonomial coefficient.

Theorem 3.3. *For $n \geq 1$,*

$$2^n \binom{n}{1}_F = \sum_{k=0}^{n-1} 2^k L_k.$$

Using our algebraic definition for the Fibonomial coefficients, this reduces to

$$2^n F_n = \sum_{j=0}^{n-1} 2^j L_j,$$

which was already proved combinatorially in Benjamin and Quinn (2003), where it is identity 236.

Proof. For a fixed n, the left side counts the number of partial circular tilings that begin at 1. We can consider the paths that begin at 1. Each must necessarily have exactly one "left" step, which could occur on any row. If the "left" step occurs on the row with j squares, then all rows below will tile 1 square, the current row will circularly tile j squares, and the remaining j rows above the current row will circularly tile 0 squares (in two ways each). This makes the total number of paths in each case $2^j L_j$. Summing over all possible rows gives $\sum_{j=0}^{n-1} 2^j L_j$. □

By taking two left steps, the same logic gives us the following.

Theorem 3.4. *For $n \geq 0$,*

$$2^n \binom{n}{2}_F = \sum_{i=0}^{n-2} \sum_{j=0}^{i} 2^j L_2^{n-i-2} L_i L_j.$$

Chapter 4

Some Analysis of the Combinatorial Interpretations

It may be surprising that for a given Fibonomial coefficient, the total number of cells tiled is constant over all the different partial tilings that correspond to that coefficient. For instance, any partial tiling for $\binom{5}{2}_F$ will always result in exactly 6 squares being tiled, as in Figure 4.1.

Through a bit of experimentation, we can discover that for a given Fibonomial coefficient $\binom{n}{k}_F$, the number of squares covered by tiles is equal to $k(n-k)$. But why should this be the case? We can investigate this more closely by reconciling the two kinds of combinatorial interpretations we have seen.

4.1 Correspondence with the Rectangular Interpretation

There are many similarities between the pair of interpretations introduced by Sagan and Savage, and the two triangle-based interpretations. So it is therefore understandable that we can find a a correspondence between these interpretations.

To establish the correspondence, consider an arbitrary partial tiling on Bennett's triangular interpretation for $\binom{n}{k}_F$. In this partial tiling, we have a path from the top of a lattice pyramid of height n, where the path has k "L" steps and $n-k$ "I" steps. The corresponding rectangle in Sagan and Savage's interpretation will be of size $(n-k) \times k$ (with $n-k$ rows and k columns). Then the lattice path through the rectangle beginning at the bottom left

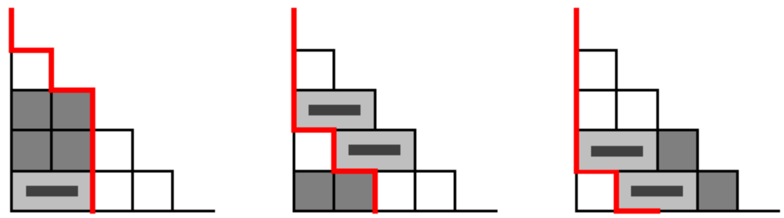


Figure 4.1 Partial Tilings of $\binom{5}{2}_F$ with exactly 6 squares tiled.

corner will correspond to the lattice path through the pyramid beginning from the top of the pyramid. For every "L" step in the pyramid, the path in they rectangle will move 1 unit to the right, and similarly, for every "I" step in the pyramid, the path through the rectangle will move 1 unit up.

So we have a clear bijection between the paths in these two structures. In fact, the number of paths in both is easily given by $\binom{n}{k}$. Now, we will also see a bijection between the tilings. Although a given lattice path may have many different tilings, it suffices to show that two corresponding paths require tiling regions of the same size.

For an arbitrary path through a pyramid, from the top to the bottom, we can characterize how many squares we need to tile at every step given only steps in the path so far. More concretely, after an "I" step, we wish to tile the squares to the left of the path. And the number of squares to tile is exactly the number of preceding "L" steps, as this is how far the path is from the left edge of the triangle. By similar logic, after an "L" step, we tile to the right, where the number of squares to tile is the number of preceding "I" steps.

Paths through a rectangle have the same property. At every "up" step, we wish to tile the squares to the left of the path. And the number of these squares is the number of preceding "right" steps. Similarly, at every "right" step, we tile the squares below the path, and the number of squares below is the number of preceding "up" steps. We even have the same restrictions on the tilings. The restriction where tilings to the right after a "L" step must begin with a domino is equivalent to the restriction where the vertical tilings in a rectangle must begin with a domino.

Thus, we have a clear correspondence between the combinatorial interpretations of Fibonomials as tilings of lattice paths through rectangles and partial tilings of paths through pyramids. One instance of the equivalence can be seen in Figure 4.2.

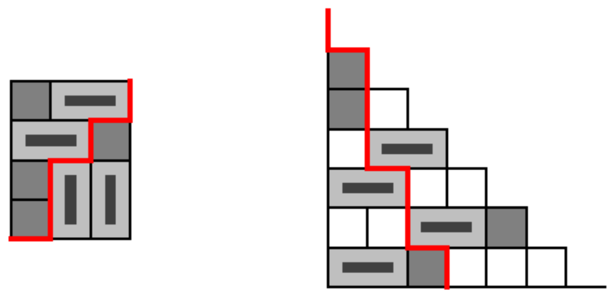


Figure 4.2 A tiled lattice path and the corresponding partial tiling.

In fact, although the above argument was for standard tilings, the same basic argument works for circular tilings. By replacing references to "I" steps with "left" steps, and "L" steps with "right" steps, we can establish the same correspondence between paths in the circularly tiled rectangle and the circularly tiled pyramid. The argument for the correspondence between tilings of these paths is even simpler than the one above, as we do not even need to consider any restrictions on what tilings are allowed.

So since there is a correspondence between the pyramid partial tilings and the rectangular tilings, it becomes obvious why all partial tilings for a particular Fibonomial coefficient all require tiling the same number of squares. As $\binom{n}{k}_F$ is also the number of tilings of lattice paths through a rectangle of size $(n-k) \times k$, and in such tilings, every square is covered by some tile, any equivalent partial tiling in a pyramid will then cover $k(n-k)$ squares, as found empirically above.

4.2 Extension Into Three Dimensions

Since we are looking at combinatorial interpretations of analogues to the binomial coefficient, it may seem natural to investigate how Fibonomials extend to these variations. In particular, here we will look at multinomial coefficients.

Recall that the multinomial coefficient is defined by

$$\binom{n}{k_1, \ldots, k_r} = \frac{n!}{k_1! \ldots k_r!}$$

where we ensure that $\sum_{i=1}^{r} k_r = n$.

Since it may be cumbersome to attempt to handle arbitrary multinomial coefficients, we will restrict our focus to those coefficients with only three elements on the bottom. (We will call these trinomial coefficients, as they give the coefficients for the expansion of a trinomial.)

Thus, it is natural to define a "Fibotrinomial coefficient" as

$$\binom{n}{a,b,c}_F = \frac{F_n F_{n-1} \dots F_1}{F_a F_{a-1} \dots F_1 F_b F_{b-1} \dots F_1 F_c F_{c-1} \dots F_1}$$

where $a+b+c=n$.

It is then trivial to show that

$$\binom{n}{a,b,c}_F = \binom{n}{a}_F \binom{n-a}{b}_F = \binom{n}{b}_F \binom{n-b}{c}_F = \binom{n}{c}_F \binom{n-c}{a}_F. \tag{4.1}$$

Furthermore, these Fibotrinomial coefficients are symmetric.

We can begin our bid to find a combinatorial interpretation by looking for a recursive formula for the Fibotrinomial coefficients.

Recall from Equation 2.1 that

$$\binom{n}{k}_F = \frac{L_k}{2}\binom{n-1}{n-k-1}_F + \frac{L_{n-k}}{2}\binom{n-1}{k-1}_F.$$

We now make heavy use of this equation to find some recursive formulae.

$$\begin{aligned}
\binom{n}{a,b,c}_F &= \binom{n}{a}_F \binom{n-a}{b}_F \\
&= \left[\frac{L_a}{2}\binom{n-1}{n-a-1}_F + \frac{L_{n-a}}{2}\binom{n-1}{a-1}_F\right]\binom{n-a}{b}_F \\
&= \frac{L_a}{2}\binom{n-1}{n-a-1}_F\binom{n-a}{b}_F + \frac{L_{n-a}}{2}\binom{n-1}{a-1}_F\binom{n-a}{b}_F \\
&= \frac{L_a}{2}\binom{n-1}{a}_F\binom{n-a}{b}_F + \frac{L_{n-a}}{2}\binom{n-1}{a-1,b,c}_F \\
&= \frac{L_a}{2}\binom{n-1}{a}_F\left[\frac{L_b}{2}\binom{n-a-1}{c-1}_F + \frac{L_c}{2}\binom{n-a-1}{b-1}_F\right] \\
&\quad + \frac{L_{n-a}}{2}\binom{n-1}{a-1,b,c}_F \\
&= \frac{L_a L_b}{4}\binom{n-1}{a,b,c-1}_F + \frac{L_a L_c}{4}\binom{n-1}{a,b-1,c}_F + \frac{L_{b+c}}{2}\binom{n-1}{a-1,b,c}_F
\end{aligned}$$

By symmetry, we can permute a, b, and c however we wish. If we want to go even deeper, we can take advantage of these arrangements with some algebra to arrive at

$$\begin{aligned}\binom{n}{a,b,c}_F = \frac{L_{b+c}}{2}\binom{n-1}{a-1,b,c}_F + \frac{L_{a+c}}{2}\binom{n-1}{a,b-1,c}_F \\ - \frac{L_{a+b}}{2}\binom{n-1}{a,b,c-1}_F + \frac{L_a L_b}{2}\binom{n-1}{a,b,c-1}_F.\end{aligned}$$

And if we assume that $a \geq b \geq c$, then we have the identity $L_a L_c - L_{a+c} = L_{a-c}$. Which reduces the equation to

$$\binom{n}{a,b,c}_F = \frac{L_{a-b}}{2}\binom{n-1}{a,b,c-1}_F + \frac{L_{b+c}}{2}\binom{n-1}{a-1,b,c}_F + \frac{L_{a+c}}{2}\binom{n-1}{a,b-1,c}_F.$$

So now we have several different ways we might express a recursive formula for a Fibotrinomial coefficient. However, if we try to use these to construct a combinatorial interpretation, the result is still rather messy and hard to visualize. Here is how it might go:

Instead of beginning with a triangular shaped lattice, we begin with a tetrahedral lattice. Looking down from the top, we see an equilateral triangle, and let's suppose that one of the corners is oriented towards the north. We can define a coordinate system where the point (a, b, c) on layer n of the tetrahedron is a units to the east, and b units to the north-east from the south-west corner of the nth layer.

Then, we can compute the value of $2^n\binom{n}{a,b,c}_F$ by considering all lattice paths from the top of the tetrahedron to the point (a, b, c) on layer n from the top. Whenever the path moves down and to the south-east, if the path is at point (x, y, z), perform a circular tiling on a $1 \times (y + z)$ grid. This is also the distance along the lattice from the path to the south-east corner of the layer. When the path moves down and to the north, we perform a circular tiling on a $1 \times (x + z)$ grid, which is the distance to the north corner. Finally, when the path moves down and to the south-west, we tile a $1 \times |x - y|$ grid. This size is equivalent to the distance along the lattice to the median line that passes through the south-west corner.

So the above interpretation is rather complicated. It is clear how it works similarly to the interpretations we have found for standard Fibonomial coefficients, but it is much more complicated. Even without considering the difficulty of visualizing and moving away from a square grid that can easily be tiled, this interpretation is less nice, because we don't get all of the symmetries we expect.

However, so that this whole venture wasn't completely useless, let's prove one simple identity.

Theorem 4.1. *For $n, a, b, c \geq 0$, $n = a + b + c$,*

$$2^n \binom{n}{a, b, c}_F = 2^n \binom{n}{b, a, c}_F.$$

Note that we did not explicitly use this kind of symmetry in our derivation of this interpretation, though we did implicitly by using Equation 4.1.

Proof. We observe that for every path and tiling through the tetrahedron to the point (a, b, c) on layer n, we can find another to point (b, a, c) on layer n by taking the mirror image across the median through the south-west corner. $\square$

If we wish to obtain other symmetries, we can adjust the coordinate system we use within the interpretation.

Chapter 5

A Brief Exploration of Alternating Sum Identities

Among the many known identities for Fibonomial coefficients, there is one class of identities that seems to appear very frequently. Two such identities,

$$\sum_{j=0}^{k}(-1)^{j(j+1)/2}\binom{k}{j}_F\binom{n-j}{k-1}_F = 0$$

and

$$\sum_{j=0}^{k}(-1)^{j(j+1)/2}\binom{k}{j}_F F_{n-j}^{k-1} = 0$$

were noted by Lind (1971). [1]

This kind of alternating sum involving Fibonomial coefficients has been studied for even longer, as a similar sum is identified by Horadam et al. (1965) as the denominator of generating functions for powers of some linear recurrences.

Some experimentation reveals that these alternating sum identities can be much more generalized. For example, we can experimentally determine that

$$\sum_{j=0}^{k}(-1)^{j(j+1)/2}\binom{k}{j}_F L_{n-j}^{k-1} = 0.$$

In fact, we might observe that instead of L_{n-j}^{k-1}, we can can have any product of the form $\prod_{i=1}^{k-1} L_{n_i-j}$, where every n_i is some integer. This is

[1]Here, I have adjusted the indices to emphasize the similarity.

reminiscent of the first identity presented in this chapter, as we recall that $\binom{n-j}{k-1}_F$ involves a product of $k-1$ Fibonacci numbers.

Some more experimenting can determine that these numbers don't even have to be Fibonacci numbers or Lucas numbers, as long as they come from a sequence satisfying the same recurrence relation. And the terms of the product don't even need to come from the same sequence. Thus, I arrived at the following result which acts as a generalization of all these alternating sum identities.

Theorem 5.1. *Given some $k \geq 1$, and sequences $\{s_1\}, \{s_2\}, \ldots, \{s_{k-1}\}$ where every sequence $\{s_i\} = \{s_{i_0}, s_{i_1}, \ldots\}$ satisfies $s_{i_n} = s_{i_{n-1}} + s_{i_{n-2}}$, then we have the identity*

$$\sum_{j=0}^{k}(-1)^{j(j+1)/2}\binom{k}{j}_F \prod_{i=1}^{k-1} s_{i_{k-j}} = 0.$$

Note that we have simply indexed into all the sequences with $k-j$ instead of $n-j$ which depends on the arbitrary parameter n. However, this formulation actually provides more freedom for the sequence indices, as we can now arbitrarily shift the indices for every sequence since there are no restrictions on initial conditions.

Lind (1971) indicates that Jarden (1966) proved some form of the above theorem, but I cannot consult the proof, so I will provide my own proof.

Before we prove this, we will need to consult some other results. First, I will rewrite the general recursive formula for the Fibonomial coefficients from Equation 1.1 in a more useful form.

$$\binom{n+1}{k}_F = F_{k-1}\binom{n}{n-k}_F + F_{n-k+2}\binom{n}{k-1}_F \tag{5.1}$$

Next, we will find it useful to be able to express any element of a sequence in terms of other terms in the sequence. In what follows, we note that we allow the integers n, k to be negative.

Lemma 5.1. For a sequence $\{a\}$ where $a_n = a_{n-1} + a_{n-2}$, for integers n, k,

$$a_n = F_{n-k-1}a_k + F_{n-k}a_{k+1}.$$

Proof. Benjamin and Quinn (2003) proved a general case of the identity

$$a_{m+n} = a_{m-1}F_n + a_mF_{n+1}$$

for positive m and n. (In their book, it is identity 73.) We can extend the equation to any n and m by simply re-indexing the sequence. Then, a simple substitution yields our desired result. $\square$

Proof of Theorem 5.1. We will prove this by induction over k. It is trivial to show that the identity holds for $k = 1$. When $k = 2$, we have a single sequence, and the identity yields exactly the recurrence relation for that sequence. With this in mind, we can just worry about cases when we have at least one sequence.

We can now attack the inductive step of the proof. So we assume that the identity holds for some k. For simplicity of notation, let us denote the sequence formed by the terms $\prod_{i=1}^{k-1} s_{i_j}$ as $\{a\}$. So then the Fibonomial coefficients $\binom{k}{j}_F$ act as coefficients for the elements a_{k-j} of this sequence.

Now let us consider another sequence $\{b\}$ that also satisfies $b_n = b_{n-1} + b_{n-2}$. We will discover a relationship between the the terms

$$a_k b_k, a_{k-1} b_{k-1}, \ldots, a_0 b_0,$$

and show that the coefficients indeed correspond to Fibonomial coefficients.

So we know

$$\sum_{j=0}^{k} (-1)^{j(j+1)/2} \binom{k}{j}_F a_{k-j} = 0,$$

and taking advantage of our freedom to reindex the sequences, we also have

$$\sum_{j=0}^{k} (-1)^{j(j+1)/2} \binom{k}{j}_F a_{k-j+1} = 0.$$

We will find our new relationship by multiplying the first equation by $(-1)^{k+1} b_0$, the second by b_{k+1}, and then summing the two equations. Now, we examine the resulting terms.

The $a_0 b_0$ term will have coefficient $(-1)^{k(k+1)/2}(-1)^{k+1} = (-1)^{(k+1)(k+2)/2}$. The $a_{k+1} b_{k+1}$ term will have coefficient 1. And for every other a_i for $0 < i < k+1$, we will have the terms

$$(-1)^{(k-i)(k-i+1)/2}(-1)^{k+1} \binom{k}{k-i}_F a_i b_0 + (-1)^{(k-i+1)(k-i+2)/2} \binom{k}{k-i+1}_F a_i b_k.$$

But incredibly, these terms simplify extremely nicely. We will use the result in Lemma 5.1 to write the left hand expression as

$$(-1)^{(k-i)(k-i+1)/2}(-1)^{k+1} \binom{k}{k-i}_F a_i (F_{-i} b_{i-1} + F_{-i+1} b_i)$$

and the right hand expression as

$$(-1)^{(k-i+1)(k-i+2)/2}\binom{k}{k-i+1}_F a_i(F_{k-i+1}b_{i-1}+F_{k-i+2}b_i).$$

where we make use of the natural extension of the Fibonacci numbers to negative indices. Then, the a_ib_{i-1} terms cancel: Since we can write F_{-i} as $(-1)^{i+1}F_i$, we can show that they must have opposite signs by multiplying together all the powers of -1 and seeing the final result is negative.

$$\begin{aligned}(-1)^{(k-i)(k-i+1)/2}(-1)^{k+1}(-1)^{i+1}(-1)^{(k-i+1)(k-i+2)/2} &= (-1)^{(k-i+1)^2+k+i+2}\\ &= (-1)^{(k-i+1)+k+i+2}\\ &= (-1)^3\end{aligned}$$

And their magnitudes are equal.

$$\begin{aligned}\left|\binom{k}{k-i}_F F_{-i}\right| &= \binom{k}{k-i}_F F_i\\ &= \frac{F_k\ldots F_1}{(F_{k-i}\ldots F_1)(F_i\ldots F_1)}F_i\\ &= \frac{F_k\ldots F_1}{(F_{k-i}\ldots F_1)(F_{i-1}\ldots F_1)}\\ &= \frac{F_k\ldots F_1}{(F_{k-i+1}\ldots F_1)(F_{i-1}\ldots F_1)}F_{k-i+1}\\ &= \binom{k}{k-i+1}_F F_{k-i+1}\end{aligned}$$

Since the a_ib_{i-1} terms cancel, we are left with only the a_ib_i terms. These terms will have the same sign, as F_{-i+1} will have the opposite sign as F_{-i}, but the rest will remain the same as in the sign calculation above. In fact, it will be exactly the sign given by

$$(-1)^{(k-i+1)(k-i+2)/2}$$

Finally, we can find the magnitude of the coefficient to the a_ib_i term by simply adding

$$\binom{k}{k-i}_F F_{i-1}+\binom{k}{k-i+1}_F F_{k-i+2}.$$

And by using the symmetric property of the Fibonomial coefficients to write $\binom{k}{k-i+1}_F$ as $\binom{k}{i-1}_F$, we get exactly the recursion formula in Equation 5.1, so the magnitude of the coefficient for $a_i b_i$ is $\binom{k+1}{k-i+1}_F$.

Now, substituting this result back into a summation, with $j = k - i + 1$, we find

$$\sum_{j=0}^{k+1} (-1)^{j(j+1)/2} \binom{k+1}{j}_F a_{k-j+1} b_{k-j+1} = 0,$$

completing the inductive step. □

Although the presentation of the above proof took more than two pages, the basic idea is very simple. So below is a concrete example to hopefully elucidate the essential argument.

Suppose we already know that squares of Lucas numbers follow the recurrence relation

$$\binom{3}{0}_F L_n^2 - \binom{3}{1}_F L_{n-1}^2 - \binom{3}{2}_F L_{n-2}^2 + \binom{3}{3}_F L_{n-3}^2 = 0.$$

We can then find a recurrence relation for the cubes of Lucas numbers by summing

$$L_0 \left(\binom{3}{0}_F L_3^2 - \binom{3}{1}_F L_2^2 - \binom{3}{2}_F L_1^2 + \binom{3}{3}_F L_0^2 \right) = 0$$

and

$$L_4 \left(\binom{3}{0}_F L_4^2 - \binom{3}{1}_F L_3^2 - \binom{3}{2}_F L_2^2 + \binom{3}{3}_F L_1^2 \right) = 0.$$

For example, to find the L_2^3 term, we consider

$$\begin{aligned} -\binom{3}{1}_F L_2^2 L_0 - \binom{3}{2}_F L_2^2 L_4 &= -\binom{3}{1}_F L_2^2 (F_{-2} L_1 + F_{-1} L_2) - \binom{3}{2}_F L_2^2 (F_2 L_1 + F_3 L_2) \\ &= -\binom{3}{1}_F F_{-1} L_2^3 - \binom{3}{2}_F F_3 L_2^3 \\ &= -\binom{4}{2}_F L_2^3. \end{aligned}$$

Here, we discover that the $L_2^2 L_1$ terms cancel, and the coefficient for the L^3 term follows from the recursive formula for the Fibonomial coefficients. We can do the same for all the other terms, and we will end with a recurrence relation for the cubes of the Lucas numbers, where the coefficients are all Fibonomial coefficients.

Chapter 6

Future Work

There are still many identities involving Fibonomial coefficients that have not been proved combinatorially. For convenience, I provide here a list of these identities, all of which were taken from a list originally compiled by Reiland (2011).

- $\binom{n}{k}_F \binom{k}{j}_F = \binom{n}{j}_F \binom{n-j}{k-j}_F$
- $\binom{n}{k}_F = \sum_{j=k}^{n} \frac{F_j - F_{j-k}}{F_k} \binom{j-1}{k-1}_F$
- $F_k \binom{n}{k}_F = F_n \binom{n-1}{k-1}_F$
- $F_k \binom{n}{k}_F = F_{n-k+1} \binom{n}{k-1}_F$
- $F_{n+1} \binom{n}{k}_F = F_{n-k+1} \binom{n+1}{k}_F$
- $\sum_{j=0}^{k} (-1)^{j(j+3)/2} \binom{k}{j}_F F_{n+k-j}^{k+1} = F_1 \ldots F_k F_{(k+1)(n+k/2)}$

- $$\sum_{j=0}^{m-1}(-1)^{j(j+3)/2}\binom{(m+1)k+m}{j}_F\binom{(m+1)k+m-j-1}{m-j-1}_F F_{n+k+m-j}^{m+1}$$
  $$+(-1)^{m(m+3)/2}F_{n-mk}^{m+1}=\left(\prod_{j=1}^{m}F_{(m+1)k+j}\right)F_{(m+1)(n+m/2)}$$

- $$\sum_{k=0}^{n}\binom{2n+1}{k}_F=\prod_{k=0}^{n}L_{2k}$$

- For odd n,
  $$\sum_{j=0}^{n}(-1)^{j(n+j)/2}\binom{n}{j}_F=0$$

- For positive integers k, $m>k$, and nonnegative integer $\ell\le\frac{k-1}{2}$,
  $$\sum_{j=0}^{m}(-1)^{j(2\ell+j+1)/2}\frac{F_{(k-j)(k-2\ell)}}{F_{k-2\ell}}\binom{k+1}{j}_F=0$$

- For positive integers k, $m>k$, and nonnegative integers $n,\ell\le\frac{k-1}{2}$,
  $$\sum_{j=0}^{m}(-1)^{j(2\ell+j+(-1)^k)/2}L_{(k-2\ell)(j_n)}\binom{k+1}{i}_F=0$$

- $$\binom{n}{k}_F-\binom{n-i}{k}_F=\binom{n-1}{k-1}_F\frac{F_n-F_k}{F_{n-k}}$$

Bibliography

Benjamin, Arthur T, and Jennifer J Quinn. 2003. *Proofs that really count: the art of combinatorial proof*. 27, MAA.

Bennett, Curtis, Juan Carrillo, John Machacek, and Bruce E Sagan. 2018. Combinatorial interpretations of lucas analogues of binomial coefficients and catalan numbers. *arXiv preprint arXiv:180909036* .

Horadam, AF, et al. 1965. Generating functions for powers of a certain generalised sequence of numbers. *Duke Mathematical Journal* 32(3):437–446.

Jarden, Dov. 1966. *Recurring sequences: A collection of papers*. Riveon Lematematika.

Lind, DA. 1971. A determinant involving generalized binomial coefficients. *Fibonacci Quart* 9(2):113–119.

Reiland, Elizabeth. 2011. Combinatorial interpretations of fibonomial identities. *HMC Senior Theses* 10.

Sagan, Bruce E, and Carla D Savage. 2010. Combinatorial interpretations of binomial coefficient analogues related to lucas sequences. *Integers* 10(6):697–703.